

# A Constraint between Noncommutative Parameters of Quantum Theories in Noncommutative Space

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## Abstract

In two-dimensional noncommutative space for the case of both position - position and momentum - momentum noncommuting, a constraint between noncommutative parameters is investigated. The related topic of guaranteeing Bose - Einstein statistics in noncommutative space in the general case are elucidated: Bose - Einstein statistics is guaranteed by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. A special character of a dynamical system is represented by a constraint between noncommutative parameters. The general feature of the constraint for any system is a direct proportionality between noncommutative parameters with a proportional coefficient depending on characteristic parameters of the system under study. The constraint for a harmonic oscillator is illustrated.

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# 1 Introduction

Physics in noncommutative space [1–3] has been extensively investigated in literature. This is motivated by studies of the low energy effective theory of D-brane with a nonzero NS - NS  $B$  field background. Effects of spatial noncommutativity are apparent only near the string scale, thus we need to work at a level of noncommutative quantum field theory. But based on the incomplete decoupling mechanism one expects that quantum mechanics in noncommutative space (NCQM) may clarify some low energy phenomenological consequences, and lead to qualitative understanding of effects of spatial noncommutativity. In literature NCQM and its applications [4–13] have been studied in detail. But an important issue about whether in noncommutative space the concept of identical particles being still meaningful and whether Bose-Einstein statistics and Fermi-Dirac statistics being still maintained has not been resolved.

On the fundamental level of quantum field theory the annihilation and creation operators appear in the expansion of the (free) field operator  $\Psi(\hat{x}) = \int d^3k a_k(t) \Phi_k(\hat{x}) + H.c.$  The consistent multi-particle interpretation requires the usual (anti)commutation relations among  $a_k$  and  $a_k^\dagger$ . Introduction of the Moyal type deformation of coordinates may yield a deformation of the algebra between the creation and annihilation operators. Various authors [14, 15] argue important consequences for Pauli's principle in the case of Fermi-Dirac statistics. In noncommutative quantum field theory Poincaré invariance is broken and is replaced by a twisted Poincaré symmetry. On the other hand this is only possible if the statistics is equally twisted [15]. Whether the deformed Heisenberg - Weyl algebra is consistent with Bose - Einstein statistics is still an open issue at the level of quantum field theory.

In this paper our study is restricted in the context of non-relativistic quantum mechanics to elucidate this problem. We follow the standard procedure of establishing Bose - Einstein statistics in the ordinary quantum mechanics in commutative space, and investigate whether Bose - Einstein statistics can be maintained in noncommutative space. For a two dimensional isotropic harmonic oscillator the consistency of the deformed Heisenberg - Weyl algebra with Bose - Einstein statistics is elucidated [9]. But this example is spe-

cial. We need to clarify the situation for general cases. We find that in the case of both noncommutativities of position - position and momentum - momentum Bose - Einstein statistics is guaranteed by the deformed Heisenberg - Weyl algebra itself, independent of dynamics. The special character of a dynamical system is represented by a relation between noncommutative parameters. The general feature of such a relation for any system is a direct proportionality between noncommutative parameters with a proportional coefficient depending on characteristic parameters of the system under study, and such a proportional coefficient can be fixed up to a dimensionless constant. The speciality of a two dimensional isotropic harmonic oscillator is that the relation between noncommutative parameters can be completely determined.

In order to demonstrate consistency between the deformed Heisenberg - Weyl algebra and Bose - Einstein statistics, in the following we first review the necessary background.

## 2 The Deformed Heisenberg - Weyl Algebra

The start point is the deformed Heisenberg - Weyl algebra. We consider the case of both position - position noncommutativity (space-time noncommutativity is not considered) and momentum - momentum noncommutativity. In this case the consistent deformed Heisenberg - Weyl algebra is [9]:

$$[\hat{x}_i, \hat{x}_j] = i\xi^2\theta\epsilon_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\xi^2\eta\epsilon_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (i, j = 1, 2), \quad (2.1)$$

where  $\theta$  and  $\eta$  are constant parameters, independent of the position and momentum. Here we consider the noncommutativity of the intrinsic canonical momentum. It means that the parameter  $\eta$ , like the parameter  $\theta$ , should be extremely small. This is guaranteed by a direct proportionality provided by a constraint between them (See Eq. (5.4) below). The  $\epsilon_{ij}$  is a two-dimensional antisymmetric unit tensor,  $\epsilon_{12} = -\epsilon_{21} = 1$ ,  $\epsilon_{11} = \epsilon_{22} = 0$ . In Eq. (2.1) the scaling factor  $\xi = (1 + \theta\eta/4\hbar^2)^{-1/2}$  is a dimensionless constant. When  $\eta = 0$ , we have  $\xi = 1$ . The deformed Heisenberg - Weyl algebra (2.1) reduces to the one of only position - position noncommuting.

In literature there is a tacit confusion about the difference between the intrinsic noncommutativity of the canonical momenta discussed here and the noncommutativity of the

mechanical momenta of a particle in an external magnetic field with a vector potential  $A_i(x_j)$  in commutative space. In the later case the mechanical momentum is

$$p_{mech,i} = \mu \dot{x}_i = p_i - \frac{q}{c} A_i, \quad (2.2)$$

where  $p_i = -i\hbar\partial_i$  is the canonical momentum in commutative space, satisfying  $[p_i, p_j] = 0$ . The commutator between  $p_{mech,i}$  and  $p_{mech,j}$  is

$$[p_{mech,i}, p_{mech,j}] = -\frac{q}{c} ([p_i, A_j] + [A_i, p_j]) = i\frac{\hbar q}{c} (\partial_i A_j - \partial_j A_i) = i\frac{\hbar q}{c} \epsilon_{ij3} B_3. \quad (2.3)$$

Such a noncommutativity is determined by the external magnetic field  $\vec{B}$  which, unlike the noncommutative parameter  $\eta$ , may not be extremely small. Thus the noncommutativity the mechanical momenta of a particle in an external magnetic field in commutative space is essentially different from the intrinsic noncommutativity, the second equation in Eq. (2.1), of the canonical momentum in noncommutative space.

The deformed Heisenberg - Weyl algebra (2.1) can be realizations by undeformed variables as follows (henceforth summation convention is used)

$$\hat{x}_i = \xi(x_i - \frac{1}{2\hbar}\theta\epsilon_{ij}p_j), \quad \hat{p}_i = \xi(p_i + \frac{1}{2\hbar}\eta\epsilon_{ij}x_j), \quad (2.4)$$

where  $x_i$  and  $p_i$  satisfy the undeformed Heisenberg - Weyl algebra  $[x_i, x_j] = [p_i, p_j] = 0$ ,  $[x_i, p_j] = i\hbar\delta_{ij}$ .

It should be emphasized that for the case of both position - position and momentum - momentum noncommuting the scaling factor  $\xi$  in Eqs. (2.1) and (2.4) guarantees consistency of the framework, and plays an essential role in dynamics. One may argues that only three parameters  $\hbar$ ,  $\theta$  and  $\eta$  can appear in three commutators (2.1), thus  $\xi$  is an additional spurious parameter and can be set to 1. If one re-scales  $\hat{x}_i$  and  $\hat{p}_i$  so that  $\xi = 1$  in Eqs. (2.1) and (2.4), it is easy to check that Eq. (2.4) leads to  $[\hat{x}_i, \hat{p}_j] = i\hbar(1 + \theta\eta/4\hbar^2)\delta_{ij}$ , thus the Heisenberg commutation relation cannot be maintained.

### 3 Consistency Between The Deformed Heisenberg - Weyl Algebra and Bose - Einstein Statistics

In noncommutative space the concept of identical particles being meaningful and Bose-Einstein statistics being maintained in the general case are elucidated by the following

theorem:

**Theorem** *In the case of both position - position and momentum - momentum noncommuting the deformed Heisenberg - Weyl algebra is consistent with Bose - Einstein statistics.*

Proving this theorem includes two aspects. The first aspect is to construct the general representations of the deformed annihilation and creation operators which satisfy the complete and closed deformed bosonic algebra [9]. The second aspect is, by generalizing one - particle quantum mechanics, to establish the Fock space of identical bosons.

In the context of quantum mechanics the general representation of the deformed annihilation and creation operator  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  by  $\hat{x}_i$  and  $\hat{p}_i$  is

$$\hat{a}_i = c_1(\hat{x}_i + ic_2\hat{p}_i), \quad \hat{a}_i^\dagger = c_1(\hat{x}_i - ic_2\hat{p}_i), \quad (3.1)$$

where  $c_1$  and  $c_2$  are constants and may depend on characteristic parameters, the mass  $\mu$ , the frequency  $\omega$  etc., of the system under study.  $c_1$  and  $c_2$  can be fixed as follows. Operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  should satisfy the bosonic commutation relations  $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1$  (to keep the physical meaning of  $\hat{a}_i$  and  $\hat{a}_i^\dagger$ ). From this requirement and the deformed Heisenberg - Weyl algebra (2.1) it follows that

$$c_1 = \sqrt{1/2\hbar c_2}. \quad (3.2)$$

Following the standard procedure in quantum mechanics, starting from a system with one particle, the state vector space of a many-particle system can be constructed by generalizing one - particle formulism. Then Bose - Einstein statistics for a identical - boson system can be developed in the standard way. Bose - Einstein statistics should be maintained at the deformed level described by  $\hat{a}_i$ , thus operators  $\hat{a}_i$  and  $\hat{a}_j$  should be commuting:  $[\hat{a}_i, \hat{a}_j] = 0$ . From this equation and the deformed Heisenberg - Weyl algebra (2.1) it follows that  $ic_1^2\xi^2\epsilon_{ij}(\theta - c_2^2\eta) = 0$ . Thus *the condition of guaranteeing Bose - Einstein statistics* reads

$$c_2 = \sqrt{\frac{\theta}{\eta}}. \quad (3.3)$$

From Eqs. (3.1), (3.2) and (3.3) we obtain the following deformed annihilation and creation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$ :

$$\hat{a}_i = \sqrt{\frac{1}{2\hbar}}\sqrt{\frac{\eta}{\theta}}\left(\hat{x}_i + i\sqrt{\frac{\theta}{\eta}}\hat{p}_i\right), \quad \hat{a}_i^\dagger = \sqrt{\frac{1}{2\hbar}}\sqrt{\frac{\eta}{\theta}}\left(\hat{x}_i - i\sqrt{\frac{\theta}{\eta}}\hat{p}_i\right), \quad (3.4)$$

From Eqs. (2.1) and (3.4) it follows that the deformed bosonic algebra of  $\hat{a}_i$  and  $\hat{a}_j^\dagger$  reads [9]

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} + \frac{i}{\hbar} \xi^2 \sqrt{\theta\eta} \epsilon_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0, \quad (i, j = 1, 2). \quad (3.5)$$

In Eqs. (3.5) the three equations  $[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1$ ,  $[\hat{a}_1, \hat{a}_2] = 0$  are the same as the undeformed bosonic algebra in commutative space; The equation

$$[\hat{a}_1, \hat{a}_2^\dagger] = \frac{i}{\hbar} \xi^2 \sqrt{\theta\eta} \quad (3.6)$$

is a new type. Eqs. (3.5) constitute a complete and closed deformed bosonic algebra. Because of noncommutativity of space, different degrees of freedom are correlated at the level of the deformed Heisenberg - Weyl algebra (1); Eq. (3.6) represents such correlations at the level of the deformed annihilation and creation operators.

Now we consider the second aspect. Following the standard procedure of constructing the Fock space of many - particle systems in commutative space, we shall take Eqs. (3.5) as the *definition relations* for the complete and closed deformed bosonic algebra without making further reference to its  $\hat{x}_i$ ,  $\hat{p}_i$  representations, generalize it to many - particle systems and find a basis of the Fock space.

We introduce the following auxiliary operators, the tilde annihilation and creation operators

$$\tilde{a}_1 = \frac{1}{\sqrt{2\alpha_1}} (\hat{a}_1 + i\hat{a}_2), \quad \tilde{a}_2 = \frac{1}{\sqrt{2\alpha_2}} (\hat{a}_1 - i\hat{a}_2), \quad (3.7)$$

where  $\alpha_{1,2} = 1 \pm \xi^2 \sqrt{\theta\eta}/\hbar$ . From Eqs. (3.5) it follows that the commutation relations of  $\tilde{a}_i$  and  $\tilde{a}_j^\dagger$  read

$$[\tilde{a}_i, \tilde{a}_j^\dagger] = \delta_{ij}, \quad [\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0, \quad (i, j = 1, 2). \quad (3.8)$$

Thus  $\tilde{a}_i$  and  $\tilde{a}_i^\dagger$  are explained as the deformed annihilation and creation operators in the tilde system. The tilde number operators  $\tilde{N}_1 = \tilde{a}_1^\dagger \tilde{a}_1$  and  $\tilde{N}_2 = \tilde{a}_2^\dagger \tilde{a}_2$  commute each other,  $[\tilde{N}_1, \tilde{N}_2] = 0$ . A general tilde state

$$|\widetilde{m, n}\rangle \equiv (m!n!)^{-1/2} (\tilde{a}_1^\dagger)^m (\tilde{a}_2^\dagger)^n |\widetilde{0, 0}\rangle, \quad (3.9)$$

where the vacuum state  $|\widetilde{0, 0}\rangle$  in the tilde system is defined as  $\tilde{a}_i |\widetilde{0, 0}\rangle = 0$  ( $i = 1, 2$ ), is the common eigenstate of  $\tilde{N}_1$  and  $\tilde{N}_2$ :  $\tilde{N}_1 |\widetilde{m, n}\rangle = m |\widetilde{m, n}\rangle$ ,  $\tilde{N}_2 |\widetilde{m, n}\rangle = n |\widetilde{m, n}\rangle$ , ( $m, n =$

$0, 1, 2, \dots$ ), and satisfies  $\langle \widehat{m'}, n' | \widehat{m}, n \rangle = \delta_{m'm} \delta_{n'n}$ . Thus  $\{|\widehat{m}, n\rangle\}$  constitute an orthogonal normalized complete basis of the tilde Fock space. In the tilde Fock space all calculations are the same as the case in commutative space, thus the concept of identical particles is maintained and the formalism of the deformed Bosonic symmetry which restricts the states under permutations of identical particles in multi - boson systems can be similarly developed.

The theorem is proved.

It should be emphasized that in the case of both position - position and momentum - momentum noncommuting the special feature is when  $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$  are satisfied, Bose - Einstein statistics is not guaranteed. The reason is as follows. Because the new type (3.6) of bosonic commutation relations correlates different degrees of freedom, the number operators  $\hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1$  and  $\hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2$  do not commute,  $[\hat{N}_1, \hat{N}_2] \neq 0$ . They have not common eigenstates. The vacuum state of the hat system is defined as  $\hat{a}_i |0, 0\rangle = 0$ , ( $i = 1, 2$ ). A general hat state  $|\widehat{m}, n\rangle$  is defined as

$$|\widehat{m}, n\rangle \equiv c(\hat{a}_1^\dagger)^m (\hat{a}_2^\dagger)^n |0, 0\rangle$$

where  $c$  is the normalization constant, these states  $|\widehat{m}, n\rangle$  are not the eigenstate of  $\hat{N}_1$  and  $\hat{N}_2$ :

$$\begin{aligned}\hat{N}_1 |\widehat{m}, n\rangle &= m |\widehat{m}, n\rangle + \frac{i}{\hbar} m \xi^2 \sqrt{\theta \eta} |m+1, n-1\rangle, \\ \hat{N}_2 |\widehat{m}, n\rangle &= n |\widehat{m}, n\rangle + \frac{i}{\hbar} n \xi^2 \sqrt{\theta \eta} |m-1, n+1\rangle.\end{aligned}$$

Because of Eq. (3.6), in calculations of the above equations we should take care of the ordering of  $a_i$  and  $a_j^\dagger$  for even  $i \neq j$  in the state  $|\widehat{m}, n\rangle$ . The states  $|\widehat{m}, n\rangle$  are not orthogonal each other. For example, the inner product between  $|\widehat{1}, 0\rangle$  and  $|\widehat{0}, 1\rangle$  is

$$\langle \widehat{1}, 0 | \widehat{1}, 0 \rangle = -\frac{i}{\hbar} \xi^2 \sqrt{\theta \eta}.$$

Thus  $\{|\widehat{m}, n\rangle\}$  do not constitute an orthogonal complete basis of the Fock space of a identical - boson system.

Now we investigate two issues related to this theorem: the tilde phase space and the constraint between noncommutative parameters.

## 4 The Tilde Phase Space

First we consider tilde phase space variables. Using Eqs. (3.4) and (3.7) we rewrite  $\tilde{a}_i$  as

$$\begin{aligned}\sqrt{\alpha_1} \tilde{a}_1 &= \left( \frac{\eta}{4\theta\hbar^2} \right)^{1/4} \left( \tilde{x} + i\sqrt{\frac{\theta}{\eta}} \tilde{p}^\dagger \right), \\ \sqrt{\alpha_2} \tilde{a}_2 &= \left( \frac{\eta}{4\theta\hbar^2} \right)^{1/4} \left( \tilde{x}^\dagger + i\sqrt{\frac{\theta}{\eta}} \tilde{p} \right).\end{aligned}\tag{4.1}$$

Where the tilde coordinate and momentum  $(\tilde{x}, \tilde{p})$  are related to  $(\hat{x}, \hat{p})$  by

$$\tilde{x} = \frac{1}{\sqrt{2}} (\hat{x}_1 + i\hat{x}_2), \quad \tilde{p} = \frac{1}{\sqrt{2}} (\hat{p}_1 - i\hat{p}_2).\tag{4.2}$$

The tilde phase variables  $(\tilde{x}, \tilde{p})$  satisfy the following commutation relations:

$$[\tilde{x}, \tilde{x}^\dagger] = \xi^2\theta, \quad [\tilde{p}, \tilde{p}^\dagger] = -\xi^2\eta, \quad [\tilde{x}, \tilde{p}] = [\tilde{x}^\dagger, \tilde{p}^\dagger] = i\hbar, \quad [\tilde{x}, \tilde{p}^\dagger] = [\tilde{x}^\dagger, \tilde{p}] = 0.\tag{4.3}$$

A Hamiltonian  $\hat{H}(\hat{x}, \hat{p}) = \hat{p}_i\hat{p}_i/2\mu + V(\hat{x}_i)$  with potential  $V(\hat{x}_i)$  in the hat system is rewritten as

$$\hat{H}(\hat{x}, \hat{p}) = \tilde{H}(\tilde{x}, \tilde{x}^\dagger, \tilde{p}, \tilde{p}^\dagger) = (\tilde{p}\tilde{p}^\dagger + \tilde{p}^\dagger\tilde{p})/2\mu + \tilde{V}(\tilde{x}, \tilde{x}^\dagger)\tag{4.4}$$

in the tilde system.

In some cases calculations in the tilde system are simpler than ones in the hat system. For example, in the hat system the Hamiltonian of a two-dimensional isotropic harmonic oscillator is  $\hat{H}(\hat{x}, \hat{p}) = \hat{p}_i\hat{p}_i/2\mu + \mu\omega^2\hat{x}_i\hat{x}_i/2$ . In the tilde system it is rewritten as

$$\tilde{H}(\tilde{x}, \tilde{x}^\dagger, \tilde{p}, \tilde{p}^\dagger) = (\tilde{p}\tilde{p}^\dagger + \tilde{p}^\dagger\tilde{p})/2\mu + \mu\omega^2(\tilde{x}\tilde{x}^\dagger + \tilde{x}^\dagger\tilde{x})/2 = \hbar(\tilde{\omega}_i\tilde{N}_i + \omega),\tag{4.5}$$

where  $\tilde{\omega}_{1,2} = \alpha_{1,2}\omega$  are effective frequencies, the tilde number operators  $\tilde{N}_1$  and  $\tilde{N}_2$  have eigenvalues  $n_1, n_2 = 0, 1, 2, \dots$ . From Eq. (4.5) it follows that the energy eigenvalues of  $\tilde{H}(\tilde{x}, \tilde{x}^\dagger, \tilde{p}, \tilde{p}^\dagger)$  are

$$\tilde{E}_{n_1, n_2} = \hbar(\tilde{\omega}_i n_i + \omega) = \hbar\omega(n_1 + n_2 + 1) + \hbar\omega\sqrt{\theta\eta}(n_1 - n_2).\tag{4.6}$$

The last term represents the shift of the energy level originated from effects of spacial noncommutativity. There is no shift for zero-point energy  $\omega$ . It is worth noting that Eq. (4.6) gives the *exact* (non-perturbational) eigenvalues.



Ref. [10] also investigated the structure of a noncommutative Fock space and obtained eigenvectors of several pairs of commuting hermitian operators which can serve as basis vectors in the noncommutative Fock space. Calculations in such a noncommutative Fock space are much complex than the above (commutative) tilde Fock space.

## 5 The Constraint Between Noncommutative Parameters

The structure of the deformed annihilation and creation operators  $\hat{a}_i$  and  $\hat{a}_i^\dagger$  in Eqs. (3.4) are determined by the deformed Heisenberg - Weyl algebra (2.1), independent of dynamics. The special character of a dynamical system is encoded in the dependence of the factor  $\sqrt{\theta/\eta}$  on characteristic parameters of the system under study. This put a constraint between  $\theta$  and  $\eta$  which can be determined as follows.

The general representation of the undeformed annihilation operator  $a_i$  by  $x_i$  and  $p_i$  is  $a_i = c'_1(x_i + ic'_2 p_i)$ , where the constants  $c'_1$  can be fixed as follows. Operators  $a_i$  and  $a_i^\dagger$  should satisfy bosonic commutation relations  $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1$ . From this requirement the undeformed Heisenberg - Weyl algebra leads to  $c'_1 = \sqrt{1/2\hbar c'_2}$ . The undeformed bosonic commutation relation  $[a_i, a_j] = 0$  is automatically satisfied, so  $c'_2$  is a free parameter. Thus the general representation of the undeformed annihilation operator reads

$$a_i = \frac{1}{\sqrt{2\hbar c'_2}}(x_i + ic'_2 p_i), \quad (5.1)$$

operators  $a_i$  and  $a_i^\dagger$  satisfy the undeformed bosonic algebra  $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$ ,  $[a_i, a_j^\dagger] = i\delta_{ij}$ .

From Eqs. (2.1), (2.4), (3.1), (3.3) and (5.1) it follows that  $\hat{a}_i$  can be represented by  $a_i$  as follows:

$$\hat{a}_i = \xi(a_i + \frac{i}{2\hbar}\sqrt{\theta\eta}\epsilon_{ij}a_j), \quad (5.2)$$

Similar to Eqs. (2.1) and (2.4), it should be emphasized that for the case of both position - position and momentum - momentum noncommuting the scaling factor  $\xi$  in Eq. (5.2) guarantees consistency of the framework. Specially, it maintains the bosonic commutation relation  $[\hat{a}_i, \hat{a}_j^\dagger] = i\delta_{ij}$ .

In the limit  $\theta, \eta \rightarrow 0$ , the deformed operators  $\hat{x}_i, \hat{p}_i, \hat{a}_i$  reduce to the undeformed ones  $x_i, p_i, a_i$ . Eq. (3.3) indicates that in this limit  $\theta/\eta$  should keep finite. From Eqs. (2.1), (2.4), (3.1), (3.3), (3.4), (5.1) and (5.2), it follows that

$$c_1 = c'_1, \quad c_2 = c'_2. \quad (5.3)$$

From Eqs. (3.3) and (5.3) we obtain the following constrained condition <sup>1</sup>

$$\eta = K\theta, \quad (5.4)$$

where the coefficient  $K = c'_2{}^{-2}$  is a constant with a dimension  $(mass/time)^2$ . Eq. (5.4) shows that the general feature of such a constraint for any system is a direct proportionality between noncommutative parameters  $\eta$  and  $\theta$ .

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<sup>1</sup> Eq. (3.4) is the most general representation of the physical annihilation operator  $\hat{a}_i$  in noncommutative space. In literature there is an extensively tacit understanding about the definition of the physical annihilation operator such that “*it is possible to construct an infinity of the creation/annihilation operators which satisfy exactly the bosonic commutation relations, but do not require any constraint on the parameters such as Eq. (5.4)*”. For example, similar to the Landau creation and annihilation operators (acting within or across Landau levels) involve mixing of spatial directions in an external magnetic field, we may define the following annihilation operator

$$\hat{a}'_i = \frac{\nu^{-1}}{\sqrt{2\hbar c'_2}} \left[ \left( \delta_{ij} - \frac{ic'_2\eta}{2\hbar} \epsilon_{ij} \right) \hat{x}_j + i \left( c'_2\delta_{ij} - \frac{i\theta}{2\hbar} \epsilon_{ij} \right) \hat{p}_j \right],$$

where  $\nu = \xi(1 - \theta\eta/4\hbar^2)$ . These operators automatically satisfy the bosonic commutation relations  $[\hat{a}'_i, \hat{a}'_j{}^\dagger] = \delta_{ij}$ ,  $[\hat{a}'_i, \hat{a}'_j] = [\hat{a}'_i{}^\dagger, \hat{a}'_j{}^\dagger] = 0$ . Moreover no constraint on the parameters  $\theta$  and  $\eta$  is required apart from the obvious one  $\eta\theta \neq 4\hbar^2$ .

The previous construction also indicates that it is not compulsory to consider both position and momentum noncommutativity. Indeed, if we take  $\eta = 0$ ,  $\nu = \xi = 1$  in the previous expression for the creation/annihilation operators, we get:

$$\hat{a}''_i = \frac{1}{\sqrt{2\hbar c'_2}} \left[ \hat{x}_i + i \left( c'_2\delta_{ij} - \frac{i\theta}{2\hbar} \epsilon_{ij} \right) \hat{p}_j \right],$$

This is also perfectly consistent.

In order to clarify the meaning of  $\hat{a}'_i$  we insert Eqs. (2.4) into it. It follows that  $[(\delta_{ij} - ic'_2\eta\epsilon_{ij}/2\hbar)\hat{x}_j + i(c'_2\delta_{ij} - i\theta\epsilon_{ij}/2\hbar)\hat{p}_j] = \xi(1 - \theta\eta/4\hbar^2)(x_i + ic'_2p_i)$ , thus  $\hat{a}'_i = (x_i + ic'_2p_i)/\sqrt{2\hbar c'_2}$ , which elucidates that  $\hat{a}'_i$  is just the undeformed annihilation operator  $a_i$  in Eq. (5.1), not the annihilation operator in noncommutative space. This explains that  $\hat{a}'_i$  and  $\hat{a}'_i{}^\dagger$  automatically satisfy the undeformed bosonic commutation relations, and no constraint on the parameters  $\theta$  and  $\eta$  is required.

For the case  $\eta = 0$ ,  $\nu = \xi = 1$ , inserting Eqs. (2.4) into  $\hat{a}''_i$ , we obtain

$$\hat{a}''_i = \frac{1}{\sqrt{2\hbar c'_2}}(x_i + ic'_2p_i),$$

which is the annihilation operator in commutative space again.

In Eq. (5.4) the proportional coefficient  $K$  is not determined. In the context of quantum mechanics for simple cases the dimensional analysis can determine  $c'_2$  up to a dimensionless constant.

As an example, we consider a harmonic oscillator. The dimension of  $c'_2$  in Eqs. (5.1) is *time/mass*. The characteristic parameters in the Hamiltonian of a harmonic oscillator are the mass  $\mu$ , frequency  $\omega$  and  $\hbar$ . The unique product of  $\mu^{t_1}$ ,  $\omega^{t_2}$  and  $\hbar^{t_3}$  possessing the dimension *time/mass* is  $\mu^{-1}\omega^{-1}$ . So one obtains  $c'_2 = \gamma/\mu\omega$ , where  $\gamma$  is a dimensionless constant and can be determined as follows.

The position  $x_i$  and momentum  $p_i$  are, respectively, represented by  $a_i$  and  $a_i^\dagger$  as

$$x_i = \sqrt{\frac{\gamma\hbar}{2\mu\omega}} (a_i + a_i^\dagger), \quad p_i = -i\sqrt{\frac{\hbar\mu\omega}{2\gamma}} (a_i - a_i^\dagger).$$

In the vacuum state  $|0\rangle$  the expectations of the kinetic and the potential energy, respectively, read

$$\overline{E_k} = \langle 0 | \frac{1}{2\mu} p_i^2 | 0 \rangle = \frac{\hbar\omega}{4\gamma}, \quad \overline{E_p} = \langle 0 | \frac{1}{2}\mu\omega^2 x_i^2 | 0 \rangle = \frac{\gamma\hbar\omega}{4}. \quad (5.5)$$

The special character of a harmonic oscillator is that in any state the expectation of the kinetic energy equals to the one of the potential energy. The condition of  $\overline{E_k} = \overline{E_p}$  leads to  $\gamma = \pm 1$ . Because of  $\overline{E_k} \geq 0$ , the only solution is  $\gamma = 1$ . Thus the constraint between  $\theta$  and  $\eta$  for a harmonic oscillator reads

$$\eta = \mu^2\omega^2\theta. \quad (5.6)$$

The method of determining such a dimensionless constant for a harmonic oscillator,  $\overline{E_k} = \overline{E_p}$ , cannot be applied to general cases. A complete determination of the proportional coefficient  $K$  in (5.4) based on fundamental principles for general cases is worth elucidating in further studies.

## 6 Discussions

We clarify the following two points to conclude the paper. (i) Bose - Einstein statistics can be investigated at two levels: the fundamental level of quantum field theory and the level of quantum mechanics. Whether the deformed Heisenberg - Weyl algebra is consistent with

Bose - Einstein statistics is still an open issue at the level of quantum field theory. Following the standard procedure of investigating Bose - Einstein statistics in quantum mechanics discussions restricted at the level of quantum mechanics are allowed and meaningful. At short distances, where spatial noncommutativity might be relevant, one also expects quantum mechanics to break down and to be replaced by noncommutative quantum field theory. But studies at the level of noncommutative quantum mechanics may explore some qualitative features of spatial noncommutativity, and some results may survive at the level of noncommutative quantum field theory. It is therefore hoped that studies at the level of noncommutative quantum mechanics may give some clue for further development. (ii) The constrained condition (5.4) is fixed by the most fundamental requirement, thus can apply to any dynamical system. Ordinary quantum mechanics is a most successful theory which has been fully confirmed by experiments. If NCQM is a realistic physics, possible modifications from NCQM to ordinary quantum mechanics should be extremely small. It means that both noncommutative parameters  $\theta$  and  $\eta$  should be extremely small. This is guaranteed by Eq. (5.4). Furthermore, it is understood that noncommutativity between positions is fundamental and the parameter  $\theta$  keeps the same for all systems. Noncommutativity between momenta arises naturally as a consequence of noncommutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the noncommutative coordinates [16]. This means that noncommutativity between momenta depends on dynamics. Thus  $\eta$  and the proportional coefficient  $K$  between  $\eta$  and  $\theta$  may depend on characteristic parameters of the Hamiltonian (or the action) of the system under study. In simple cases when dimensional analysis works, it can determine  $K$  up to a dimensionless constant. In order to completely fix  $K$  considerations from dynamics may be necessary.

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